

Hardy inequalities on Riemannian manifolds and applications

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April 6, 2012

Abstract

We prove a simple sufficient criteria to obtain some Hardy inequalities on Riemannian manifolds related to quasilinear second-order differential operator $\Delta_p u := \operatorname{div} (|\nabla u|^{p-2} \nabla u)$. Namely, if ρ is a nonnegative weight such that $-\Delta_p \rho \geq 0$, then the Hardy inequality

$$c \int_M \frac{|u|^p}{\rho^p} |\nabla \rho|^p dv_g \leq \int_M |\nabla u|^p dv_g, \quad u \in C_0^\infty(M).$$

holds. We show concrete examples specializing the function ρ .

Our approach allows to obtain a characterization of p -hyperbolic manifolds as well as other inequalities related to Caccioppoli inequalities, weighted Gagliardo-Nirenberg inequalities, uncertain principle and first order Caffarelli-Kohn-Nirenberg interpolation inequality.

Keywords. Hardy inequality, Riemannian manifolds, parabolic manifolds, Caccioppoli inequality, weighted Gagliardo-Nirenberg inequality, interpolation inequality.

MSC 2010. 58J05, 31C12, 26D10

1 Introduction

An N -dimensional generalization of the classical Hardy inequality asserts that for every $p > 1$

$$c \int_\Omega \frac{|u|^p}{w^p} dx \leq \int_\Omega |\nabla u|^p dx, \quad u \in C_0^\infty(\Omega),$$

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where $\Omega \subset \mathbb{R}^N$ is an open set, and the weight w is, for instance, $w(x) := |x|$ and $p < N$, or $w(x) := \text{dist}(x, \partial\Omega)$ and Ω is convex (see for example [BFT, BM, DH, GGM, MMP, MS1, MS2, Mi] and references therein).

The preeminent role of Hardy inequalities and the knowledge of the best constants involved is a well known fact, as the reader can recognize from the wide literature that uses such a tool in Euclidean or in subelliptic setting as well as on manifolds ([BG, BC, BM, BD, Ja, MP] just to cite a few).

On the other hand, the knowledge of the validity of a Hardy or Gagliardo-Nirenberg or Sobolev or Caffarelli-Kohn-Nirenberg inequality on a manifold M and their best constants allows to obtain qualitative properties on the manifold M . For instance in [AX, CX, Xi] it was shown that if M is a complete open Riemannian manifold with nonnegative Ricci curvature in which a Hardy or Gagliardo-Nirenberg or Caffarelli-Kohn-Nirenberg type inequality holds, then M is in some suitable sense *close* to the Euclidean space.

One of our aims is to prove some Hardy inequalities on Riemannian manifolds. In 1997, Carron in [Ca2] studies weighted L^2 -Hardy inequalities on a Riemannian manifold M under some geometric assumptions on the weight function ρ , obtaining, among other results, the following inequality

$$c \int_M \frac{u^2}{\rho^2} dv_g \leq \int_M |\nabla u|^2 dv_g, \quad u \in C_0^\infty(M), \quad (1.1)$$

where ρ is a nonnegative function such that $|\nabla \rho| = 1$, $\Delta \rho \geq \frac{\gamma}{\rho}$, $\rho^{-1}\{0\}$ is a compact set of zero capacity and $c = \left(\frac{\gamma-1}{2}\right)^2$.

Li and Wang in [LW] prove that if M is a hyperbolic manifold (i.e. there exists a symmetric positive Green function $G_x(\cdot)$ for the Laplacian with pole at x), then

$$\frac{1}{4} \int_M \frac{|\nabla_y G_x(y)|^2}{G_x^2(y)} u^2(y) dv_g \leq \int_M |\nabla u|^2(y) dv_g, \quad u \in C_0^\infty(M \setminus \{x\}).$$

We also mention Miklyukov and Vuorinen, which in [MV] prove that the inequality

$$\left(\int_M |\alpha(\epsilon(x)) u(x)|^q dv_g \right)^{1/q} \leq \lambda \left(\int_M (\beta(\epsilon(x)) |\nabla u(x)|)^p dv_g \right)^{1/p}, \quad u \in W_0^{1,p}(M),$$

holds for $q \geq p$ provided some conditions related to the isoperimetric profile of M are satisfied.

In [AS], Adimurthi and Sekar use the fundamental solution of a general second-order elliptic operator to derive Hardy-type inequalities and then they extend their arguments to Riemannian manifolds using the fundamental solution of p -Laplacian.

Bozhkov and Mitidieri in [BoMi1] prove the validity of (1.1) also for $p \neq 2$ ($1 < p < N$), provided there exists on M a C^1 conformal Killing vector field K such that $\text{div } K = \mu$ with μ a positive constant and $\rho = |K|$.

Let $p > 1$ and let ρ be a nonnegative function. Our principal result is a simple criterion to establish if there holds a Hardy inequality involving the weight ρ . Namely, if ρ is p -superharmonic in Ω , that is $-\Delta_p \rho \geq 0$, then

$$c \int_M \frac{|u|^p}{\rho^p} |\nabla \rho|^p dv_g \leq \int_M |\nabla u|^p dv_g, \quad u \in C_0^\infty(M), \quad (1.2)$$

holds (see Theorem 2.1). Such a kind of criteria is already established in [Da3] for a quite general class of second order operators containing, among other examples, the subelliptic operators on Carnot groups. For this goal we shall mainly use a technique introduced by Mitidieri in [Mi] and developed in [Da1, Da2, Da3] and in [BoMi1, BoMi2]. The proof is based on the divergence theorem and on the careful choice of a vector field.

Let us point out some interesting outcomes of our approach. A first issue is that, in concrete cases, our result yields an explicit value of the constant c . Moreover, in several cases, this value is also the best constant (see [Da3]). Furthermore, our approach includes Hardy inequalities already studied in [AS, BoMi1, Ca2, LW] in the case $p = 2$ as well as their generalization to $p > 1$.

Another aspect of our technique is that it allows to characterize the p -hyperbolic manifolds. We remind that a manifold M is called p -hyperbolic if there exists a symmetric positive Green function $G_x(\cdot)$ for the p -Laplacian with pole at x . We prove that M is p -hyperbolic if and only if there exists a nonnegative non trivial function $f \in L_{loc}^1(M)$ such that

$$\int_M f |u|^p dv_g \leq \int_M |\nabla u|^p dv_g, \quad u \in C_0^\infty(M).$$

Notice that one of the implications of this characterization for $p = 2$ is the result proved in [LW]. During the review process of this work, we receive the paper of Devyver, Fraas and Pinchover [DFP]. In [DFP] a general linear second order differential operator P in the Euclidean framework is studied. The authors find a profound relation between the existence of positive supersolutions of $Pu = 0$, Hardy type inequalities involving P and a weight W and the characterization of the spectrum of the weighted operator. We refer the interested reader to [DFP, DFP2].

We also obtain a generalization of (1.2). Namely, for a nonnegative function ρ , the inequality

$$\left(\frac{|p-1-\alpha|}{p} \right)^p \int_M \frac{|u|^p}{\rho^p} \rho^\alpha |\nabla \rho|^p dv_g \leq \int_M |\nabla u|^p \rho^\alpha dv_g, \quad u \in C_0^\infty(M), \quad (1.3)$$

holds, provided $-(p-1-\alpha)\Delta_p \rho \geq 0$ (see Theorem 3.1). The above inequality contains, as special case, the Caccioppoli inequality. Indeed, if ρ is a p -subharmonic function, that is $\Delta_p \rho \geq 0$, then (1.3) holds for $\alpha > p-1$ and, in particular, for $\alpha = p$ we have

$$\frac{1}{p^p} \int_M |u|^p |\nabla \rho|^p dv_g \leq \int_M |\nabla u|^p \rho^p dv_g, \quad u \in C_0^\infty(M).$$

This is the so called Caccioppoli inequality (see for instance [PRS] and the references therein for the version $p = 2$ on manifolds).

Another advantage of our approach is that it allows to obtain also other new and known results, like wighted Gagliardo-Nirenberg inequalities and the uncertain principle.

Finally we show that if (1.3) and a Sobolev type inequality (that is $c|u|_{L^{p^*}} \leq |\nabla u|_{L^p}$) hold on M , then we obtain an interpolation inequality involving, as weights, ρ and its gradient. As particular case, our results contain inequalities on manifolds related to the celebrated Caffarelli-Kohn-Nirenberg inequality.

The paper is organized as follows. We present the proof of (1.2) in Section 2, where important consequences and observations are derived. In Section 3 we show natural extensions of (1.2), obtaining also Hardy inequality with weights, Caccioppoli-type inequalities, weighted Gagliardo-Nirenberg inequalities and the uncertain principle. Some remarks on the best constant and if it is attained are discussed in Section 4. In Section 5 we present a first order interpolation inequality. Finally Section 6 is devoted to present some concrete examples of Hardy-type inequalities on manifolds.

Notation

In what follows (M, g) is a complete Riemannian N -dimensional manifold, $\Omega \subset M$ is an open set, dv_g is the volume form associated to the metric g , ∇u and $\operatorname{div} h$ stand respectively for the gradient of a function u and the divergence of a vector field h with respect to the metric g (see [Au] for further details). Throughout this paper $p > 1$.

2 Hardy inequalities

In order to state Hardy inequalities involving a weight ρ , the basic assumption we made on ρ is that ρ is p -superharmonic in weak sense. Namely, we assume that $\rho \in L^1_{loc}(\Omega)$, $|\nabla \rho| \in L^{p-1}_{loc}(\Omega)$, and $-\Delta_p \rho \geq 0$ on Ω in weak sense, that is for every nonnegative $\varphi \in C^1_0(\Omega)$, we have

$$\int_{\Omega} |\nabla \rho|^{p-2} (\nabla \rho \cdot \nabla \varphi) dv_g \geq 0. \quad (2.4)$$

The main result on Hardy inequalities is the following:

Theorem 2.1. *Let $\rho \in W^{1,p}_{loc}(\Omega)$ be a nonnegative function on Ω such that*

$$-\Delta_p \rho \geq 0 \quad \text{on } \Omega \text{ in weak sense.}$$

Then $\frac{|\nabla \rho|^p}{\rho^p} \in L^1_{loc}(\Omega)$, and the following inequality holds:

$$\left(\frac{p-1}{p}\right)^p \int_{\Omega} \frac{|u|^p}{\rho^p} |\nabla \rho|^p dv_g \leq \int_{\Omega} |\nabla u|^p dv_g, \quad u \in C^\infty_0(\Omega). \quad (2.5)$$

Before proving Theorem 2.1, we shall present some immediate consequences and extensions of the main result.

Definition 2.2. Let $\Omega \subset M$ be an open set. We denote by $D^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$|u|_{D^{1,p}} = \left(\int_{\Omega} |\nabla u|^p dv_g \right)^{1/p}.$$

It is possible to extend the validity of (2.5) to function $u \in C_0^\infty(M)$. This extension is based on the inclusion

$$D^{1,p}(M) \subset D^{1,p}(\Omega). \quad (2.6)$$

The above inclusion is satisfied, for instance, when $M \setminus \Omega$ is a compact set of zero p -capacity (see Appendix A).

Corollary 2.3. Let $\rho \in L_{loc}^1(M)$ be a function satisfying the assumptions of Theorem 2.1. If (2.6) holds, then $\frac{|\nabla \rho|^p}{\rho^p} \in L_{loc}^1(M)$, and the following inequality holds:

$$\left(\frac{p-1}{p} \right)^p \int_M \frac{|u|^p}{\rho^p} |\nabla \rho|^p dv_g \leq \int_M |\nabla u|^p dv_g, \quad u \in C_0^\infty(M). \quad (2.7)$$

Proof. The inequality (2.5) holds for every $u \in C_0^\infty(\Omega)$, then it holds for every $u \in D^{1,p}(\Omega)$. Since $C_0^\infty(M) \subset D^{1,p}(M)$, by using (2.6) we conclude the proof. \square

In order to illustrate further consequences of Theorem 2.1 we give the following:

Definition 2.4. A manifold M is said p -hyperbolic¹ if there exists a symmetric positive Green function $G_x(\cdot)$ for the p -Laplacian with pole at x^2 , if it is not the case we call it p -parabolic.

Several equivalent definitions of p -parabolic manifolds can be given. For instance in [Tr1] there is the following (see also the literature therein and [Ho2])

Proposition 2.5. Let $p > 1$. The following statements are equivalent

- a) M is p -parabolic;
- b) there exists a compact set $K \subset M$ with non empty interior such that $\text{cap}_p(K, M) = 0$;
- c) there is no non constant positive p -superharmonic function on M ;
- d) there exists a sequence of functions $u_j \in C_0^\infty(M)$ such that $0 \leq u_j \leq 1$, $u_j \rightarrow 1$ uniformly on every compact subset of M and $\int_M |\nabla u_j|^p dv_g \rightarrow 0$.

¹Many authors call these manifolds non p -parabolic.

² That is $-\Delta_p G_x = \delta_x$ where δ_x is the Dirac measure concentrated at point x .

Other characterizations of p -parabolic manifolds are based on several properties, for instance on the volume growth, on the isoperimetric profile of the manifold, on some properties of some cohomology, on the recurrence of the Brownian motion. See [Gr1, Gr2, Gr3, Gr4, LT, Tr1] and the references therein.

From Theorem 2.1 we deduce the following characterization of p -hyperbolicity.

Theorem 2.6. *A manifold M is p -hyperbolic if and only if there exists a nonnegative non trivial function $f \in L^1_{loc}(M)$ such that*

$$\int_M f |u|^p dv_g \leq \int_M |\nabla u|^p dv_g, \quad u \in C_0^\infty(M). \quad (2.8)$$

Proof. If M is p -hyperbolic then inequality (2.8) holds with $f = \left(\frac{p-1}{p}\right)^p \frac{|\nabla G_x|^p}{G_x^p}$. Indeed G_x is nonnegative and satisfies hypotheses of Theorem 2.1 (see Theorem 6.4 for further details).

Conversely, assume that M is p -parabolic and that (2.8) is valid for a function $f \geq 0$. Then from $d)$ of Proposition 2.5 there exists a sequence of functions $u_j \in C_0^1(M)$ such that $0 \leq u_j \leq 1$, $u_j \rightarrow 1$ uniformly on every compact subset of M and

$$\int_M |\nabla u_j|^p dv_g \rightarrow 0, \quad (\text{as } j \rightarrow +\infty).$$

It implies that $\int_D f dv_g = 0$ for every compact subset D of M and then $f \equiv 0$. This concludes the proof. \square

Remark 2.7. Since the p -hyperbolicity of M is equivalent to the existence of a nonconstant positive p -superharmonic function ρ on M , then by Theorem 2.1 we obtain that inequality (2.8) holds with $f = \left(\frac{p-1}{p}\right)^p \frac{|\nabla \rho|^p}{\rho^p}$.

Remark 2.8. Our Theorem 2.6 implies that if the manifold M admits a C^1 conformal Killing vector field K (see i.e. [BoMi1] for the definition) such that $\operatorname{div} K = \mu \neq 0$ with μ constant and $|K|^{-p} \in L^1_{loc}(M)$, then M is p -hyperbolic. This follows combining Theorem 2.6 and Theorem 4 of [BoMi1] (see also Remark 2.11 ii) below).

In order to prove Theorem 2.1 we fix some notation. Let $h \in L^1_{loc}(\Omega)$ be a vector field. We remind that the distribution $\operatorname{div} h$ is defined as

$$\int_\Omega \varphi \operatorname{div} h dv_g = - \int_\Omega (\nabla \varphi \cdot h) dv_g, \quad (2.9)$$

for every $\varphi \in C_0^1(\Omega)$.

Let $h \in L_{loc}^1(\Omega)$ be a vector field and let $A \in L_{loc}^1(\Omega)$ be a function. In what follows we write $A \leq \operatorname{div} h$ meaning that the inequality holds in distributional sense, that is for every $\varphi \in C_0^1(\Omega)$ such that $\varphi \geq 0$, we have

$$\int_{\Omega} \varphi A \, dv_g \leq \int_{\Omega} \varphi \operatorname{div} h \, dv_g = - \int_{\Omega} (\nabla \varphi \cdot h) \, dv_g. \quad (2.10)$$

Remark 2.9. Let $f \in C^1(\mathbb{R})$ be a real function such that $f(0) = 0$. Taking $\varphi = f(u)$ with $u \in C_0^1(\Omega)$ in (2.9), we have

$$\int_{\Omega} f(u) \operatorname{div} h \, dv_g = - \int_{\Omega} f'(u) (\nabla u \cdot h) \, dv_g.$$

In particular, choosing $f(u) = |u|^p$, with $p > 1$, we get

$$\int_{\Omega} |u|^p \operatorname{div} h \, dv_g = -p \int_{\Omega} |u|^{p-2} u (\nabla u \cdot h) \, dv_g, \quad u \in C_0^1(\Omega). \quad (2.11)$$

Lemma 2.10. Let $h \in L_{loc}^1(\Omega)$ be a vector field and let $A_h \in L_{loc}^1(\Omega)$ be a nonnegative function such that

- i) $A_h \leq \operatorname{div} h$,
- ii) $\frac{|h|^p}{A_h^{p-1}} \in L_{loc}^1(\Omega)$.

Then for every $u \in C_0^1(\Omega)$ we have

$$\int_{\Omega} |u|^p A_h \, dv_g \leq p^p \int_{\Omega} \frac{|h|^p}{A_h^{p-1}} |\nabla u|^p \, dv_g. \quad (2.12)$$

Proof. We note that the right hand side of (2.12) is finite since $u \in C_0^1(\Omega)$. Using the identity (2.11) and Hölder inequality we obtain

$$\begin{aligned} \int_{\Omega} |u|^p A_h \, dv_g &\leq \int_{\Omega} |u|^p \operatorname{div} h \, dv_g \\ &\leq p \int_{\Omega} |u|^{p-1} |h| |\nabla u| \, dv_g \\ &= p \int_{\Omega} |u|^{p-1} A_h^{(p-1)/p} \frac{|h|}{A_h^{(p-1)/p}} |\nabla u| \, dv_g \\ &\leq p \left(\int_{\Omega} |u|^p A_h \, dv_g \right)^{(p-1)/p} \left(\int_{\Omega} \frac{|h|^p}{A_h^{p-1}} |\nabla u|^p \, dv_g \right)^{1/p}. \end{aligned}$$

This completes the proof. \square

Specializing the vector field h and the function A_h , we shall deduce from (2.12) Hardy-type inequalities on Riemannian manifolds.

Remark 2.11. Letting us to point out a strategy to get Hardy inequalities at least in some special cases. Under the hypotheses of Lemma 2.10, if $A_h = \operatorname{div} h$, then (2.12) reads as

$$\int_{\Omega} |u|^p \operatorname{div} h \, dv_g \leq p^p \int_{\Omega} \frac{|h|^p}{|\operatorname{div} h|^{p-1}} |\nabla u|^p \, dv_g. \quad (2.13)$$

- i) Let V be a function in $L_{loc}^1(\Omega)$ such that its weak partial derivatives of order up to two are in $L_{loc}^1(\Omega)$. If $\Delta V \geq 0$, choosing $h = \nabla V$, we obtain $A_h = \operatorname{div} h = \operatorname{div}(\nabla V) = \Delta V \geq 0$. Then from (2.13)

$$\int_{\Omega} |u|^p |\Delta V| \, dv_g \leq p^p \int_{\Omega} \frac{|\nabla V|^p}{|\Delta V|^{p-1}} |\nabla u|^p \, dv_g. \quad (2.14)$$

This kind of inequalities for the Euclidean setting $\Omega = \mathbb{R}^N$ are already found by Davies e Hinz in [DH].

At this point, in order to deduce from (2.14) an inequality like

$$c \int_{\Omega} \frac{|u|^p}{\rho^p} \, dv_g \leq \int_{\Omega} |\nabla u|^p \, dv_g, \quad (2.15)$$

we have to choose a suitable function V . Let us to consider the case when ρ is the distance from a point $q \in M$ and $|\nabla \rho| = 1$. A suitable choice for V is $V = \rho^{2-p}$ if $1 < p < 2$, $V = \ln \rho$ if $p = 2$ and $V = -\rho^{2-p}$ if $2 < p < N$. Following [Ca2], if we require that $\Delta \rho \geq \frac{\gamma}{\rho}$ the above choices yield the inequality (2.15) with $c = \left(\frac{\gamma-p+1}{p}\right)^p$. The success of this strategy is deeply linked to the hypothesis $|\nabla \rho| = 1$. Indeed, it seems that such a strategy does not work even in the subelliptic setting, where the analogous of the hypothesis $|\nabla \rho| = 1$ does not hold.

- ii) Let $p < N$. Assume that there exists a C^1 conformal Killing vector field K (see i.e. [BoMil] for the definition) such that $\operatorname{div} K = \frac{N}{2}\mu > 0$. Choosing $h = \frac{K}{|K|^p}$, we have $A_h := \operatorname{div} h = \frac{N-p}{2} \frac{\mu}{|K|^p}$ (see Lemma 3 in [BoMil]) and the inequality (2.13) reads as

$$\left(\frac{N-p}{Np}\right)^p \int_{\Omega} \frac{\operatorname{div} K}{|K|^p} |u|^p \, dv_g \leq \int_{\Omega} (\operatorname{div} K)^{1-p} |\nabla u|^p \, dv_g. \quad (2.16)$$

Therefore, by Lemma 2.10, (2.16) holds for every $u \in C_0^1(\Omega)$ provided $|K|^{-p} \in L_{loc}^1(\Omega)$. This last fact was obtained in [BoMil, Theorem 4].

Proof of Theorem 2.1. Let $0 < \delta < 1$, and $\rho_{\delta} := \rho + \delta$. In order to apply Lemma 2.10, we define h and A_h as

$$h := -\frac{|\nabla \rho_{\delta}|^{p-2} \nabla \rho_{\delta}}{\rho_{\delta}^{p-1}} \quad \text{and} \quad A_h := (p-1) \frac{|\nabla \rho_{\delta}|^p}{\rho_{\delta}^p}. \quad (2.17)$$

Since $\frac{1}{\rho_\delta} \leq \frac{1}{\delta}$, the fact that $\rho \in W_{loc}^{1,p}(\Omega)$ implies that $h \in L_{loc}^1(\Omega)$ and $A_h \in L_{loc}^1(\Omega)$. Moreover, by computation we have

$$\frac{|h|^p}{A_h^{p-1}} = \frac{|\nabla \rho_\delta|^{p(p-1)}}{\rho_\delta^{p(p-1)}} \frac{\rho_\delta^{p(p-1)}}{|\nabla \rho_\delta|^{p(p-1)}} \frac{1}{(p-1)^{(p-1)}} = \frac{1}{(p-1)^{p-1}} \in L_{loc}^1(\Omega),$$

that is *ii*) of Lemma 2.10 is fulfilled.

The hypothesis *i*) of Lemma 2.10 is satisfied provided

$$(p-1) \int_{\Omega} \frac{|\nabla \rho_\delta|^p}{\rho_\delta^p} \varphi \, dv_g \leq \int_{\Omega} \left(\frac{|\nabla \rho_\delta|^{p-2} \nabla \rho_\delta}{\rho_\delta^{p-1}} \cdot \nabla \varphi \right) dv_g \quad (2.18)$$

holds for every nonnegative function $\varphi \in C_0^1(\Omega)$. Then, for a fixed $\varphi \in C_0^1(\Omega)$ nonnegative, we have to prove (2.18). Let $K = \text{supp} \varphi \subset \Omega$ and let U be a neighborhood of K with compact closure in Ω . We note that both integrals in (2.18) are finite since $\frac{1}{\rho_\delta} \leq \frac{1}{\delta}$ and $\rho \in W_{loc}^{1,p}(\Omega)$. Since

$$|\nabla \ln \rho_\delta| = \frac{|\nabla \rho_\delta|}{\rho_\delta} \leq \frac{|\nabla \rho|}{\delta} \in L_{loc}^p(\Omega), \quad (2.19)$$

and $\ln \rho_\delta \in L_{loc}^p(\Omega)$, we have that $\ln \rho_\delta \in W^{1,p}(U)$. Thus, for every $n \in \mathbb{N}$ there exists $\phi_n \in C^\infty(U)$ such that $|\phi_n - \ln \rho_\delta|_{W^{1,p}} < 1/n$, $\phi_n \rightarrow \ln \rho_\delta$ pointwise a.e. and $\ln \delta \leq \phi_n$ ³.

Setting $\psi_n = e^{\phi_n}$ we have that $\psi_n \in C^\infty(U)$, $\delta \leq \psi_n$, $\psi_n \rightarrow \rho_\delta$ a.e. and

$$\int_K |\ln \psi_n - \ln \rho_\delta|^p \, dv_g \rightarrow 0, \quad \int_K \left| \frac{\nabla \psi_n}{\psi_n} - \frac{\nabla \rho_\delta}{\rho_\delta} \right|^p \, dv_g \rightarrow 0, \quad (\text{as } n \rightarrow 0 + \infty). \quad (2.20)$$

For every $n \in \mathbb{N}$, the function φ_n defined as $\varphi_n := \frac{\varphi}{\psi_n^{p-1}}$ belongs to $C_0^1(\Omega)$ and it is nonnegative since $\varphi \in C_0^1(\Omega)$ is nonnegative and $\psi_n > 0$. Using φ_n as test function in (2.4) we have

$$0 \leq \int_{\Omega} |\nabla \rho|^{p-2} (\nabla \rho \cdot \nabla \varphi_n) \, dv_g = \int_{\Omega} |\nabla \rho|^{p-2} \left(\nabla \rho \cdot \nabla \left(\frac{\varphi}{\psi_n^{p-1}} \right) \right) \, dv_g, \quad (2.21)$$

³Reminding that the Sobolev space $W^{1,p}(\Omega)$ is the completion of the set

$$\left\{ u \in C^\infty(\Omega) : \int_{\Omega} |u|^p \, dv_g < \infty \text{ and } \int_{\Omega} |\nabla u|^p \, dv_g < \infty \right\}$$

with respect to the norm

$$|u|_{W^{1,p}} = \left(\int_{\Omega} |u|^p \, dv_g + \int_{\Omega} |\nabla u|^p \, dv_g \right)^{1/p},$$

the approximation result follows by slight modification of classical arguments that the reader can find, for instance, in [Au].

which, since by computation $\nabla \left(\frac{\varphi}{\psi_n^{p-1}} \right) = \frac{\nabla \varphi}{\psi_n^{p-1}} - (p-1) \frac{\nabla \psi_n}{\psi_n^p} \varphi$, implies

$$(p-1) \int_{\Omega} \frac{|\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \psi_n}{\psi_n^p} \varphi \, dv_g \leq \int_{\Omega} \left(\frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1}} \cdot \nabla \varphi \right) dv_g. \quad (2.22)$$

Now, letting $n \rightarrow +\infty$ we obtain by dominated convergence:

$$\int_{\Omega} \left(\frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1}} \cdot \nabla \varphi \right) dv_g \longrightarrow \int_{\Omega} \left(\frac{|\nabla \rho|^{p-2} \nabla \rho}{\rho_{\delta}^{p-1}} \cdot \nabla \varphi \right) dv_g,$$

because $\left| \frac{|\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \varphi}{\psi_n^{p-1}} \right| \leq \frac{C |\nabla \rho|^{p-1}}{\delta^{p-1}} \in L^1(U)$. Now we claim that

$$\int_{\Omega} \frac{|\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \psi_n}{\psi_n^p} \varphi \, dv_g = \int_{\Omega} \frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1}} \frac{\nabla \psi_n}{\psi_n} \varphi \, dv_g \longrightarrow \int_{\Omega} \frac{|\nabla \rho|^p}{\rho_{\delta}^p} \varphi \, dv_g.$$

Indeed,

$$\frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1}} \rightarrow \frac{|\nabla \rho|^{p-2} \nabla \rho}{\rho_{\delta}^{p-1}} \quad \text{pointwise a. e.}$$

and, since $\frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1}} \leq \frac{|\nabla \rho|^{p-1}}{\delta^{p-1}} \in L^{p'}(U)$, by Lebesgue dominated convergence theorem we have that

$$\frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1}} \rightarrow \frac{|\nabla \rho|^{p-2} \nabla \rho}{\rho_{\delta}^{p-1}} \quad \text{in } L^{p'}(U).$$

From this and the fact that

$$\frac{\nabla \psi_n}{\psi_n} \rightarrow \frac{\nabla \rho}{\rho_{\delta}} \quad \text{in } L^p(U)$$

we get the claim. Therefore, letting $n \rightarrow +\infty$ in (2.22), we have

$$(p-1) \int_{\Omega} \frac{|\nabla \rho|^p}{\rho_{\delta}^p} \varphi \, dv_g \leq \int_{\Omega} \left(\frac{|\nabla \rho|^{p-2} \nabla \rho}{\rho_{\delta}^{p-1}} \cdot \nabla \varphi \right) dv_g,$$

which is exactly (2.18), since $\nabla \rho_{\delta} = \nabla \rho$.

An application of Lemma 2.10 gives

$$\left(\frac{p-1}{p} \right)^p \int_{\Omega} \frac{|u|^p}{\rho_{\delta}^p} |\nabla \rho_{\delta}|^p \, dv_g \leq \int_{\Omega} |\nabla u|^p \, dv_g. \quad (2.23)$$

Finally, letting $\delta \rightarrow 0$ in (2.23) and using Fatou's Lemma, we conclude the proof. \square

3 Further inequalities

In this section we shall present some slight but natural extensions of Theorem 2.1 and Lemma 2.10. As byproducts of these generalizations we shall obtain Hardy inequalities with a weight in the right hand side, Caccioppoli-type inequalities, weighted Gagliardo-Nirenberg inequalities and the uncertain principle.

A first example of a possible generalization of Theorem 2.1 is the following:

Theorem 3.1. *Let $\alpha \in \mathbb{R}$, and let $\rho \in W_{loc}^{1,p}(\Omega)$ be a nonnegative function satisfying the following properties:*

- i) $-(p-1-\alpha) \Delta_p \rho \geq 0$ on Ω in weak sense,
- ii) $\frac{|\nabla \rho|^p}{\rho^{p-\alpha}}, \rho^\alpha \in L_{loc}^1(\Omega)$.

Then the following Hardy inequality holds

$$\left(\frac{|p-1-\alpha|}{p} \right)^p \int_{\Omega} \rho^\alpha \frac{|u|^p}{\rho^p} |\nabla \rho|^p dv_g \leq \int_{\Omega} \rho^\alpha |\nabla u|^p dv_g, \quad u \in C_0^\infty(\Omega). \quad (3.24)$$

The proof of the above theorem is similar to the one of Theorem 2.1 and it is based on a careful choice of the vector field h and of the function A_h in Lemma 2.10.

Proof. Let $0 < \delta < 1$, and $\rho_\delta := \rho + \delta$. In order to apply Lemma 2.10 we choose the vector field h and the function A_h as

$$h := -(p-1-\alpha) \frac{|\nabla \rho_\delta|^{p-2} \nabla \rho_\delta}{\rho_\delta^{p-1-\alpha}}, \quad A_h := (p-1-\alpha)^2 \frac{|\nabla \rho_\delta|^p}{\rho_\delta^{p-\alpha}}. \quad (3.25)$$

Arguing as in the proof of Theorem 2.1, we have to show that

$$(p-1-\alpha)^2 \int_{\Omega} \frac{|\nabla \rho_\delta|^p}{\rho_\delta^{p-\alpha}} \varphi dv_g \leq (p-1-\alpha) \int_{\Omega} \left(\frac{|\nabla \rho_\delta|^{p-2} \nabla \rho_\delta}{\rho_\delta^{p-1-\alpha}} \cdot \nabla \varphi \right) dv_g, \quad (3.26)$$

for every nonnegative function $\varphi \in C_0^1(\Omega)$. Let $K := \text{supp} \varphi \subset \Omega$ and let $U \subset\subset \Omega$ be a neighborhood of K . Let $k > \delta$, and define $\rho_{k\delta} := \inf \{\rho_\delta, k\}$. Arguing as in the proof of Theorem 2.1, we have that there exists a sequence $\{\psi_n\}$ such that $\delta \leq \psi_n \leq k$, and

$$\int_K |\ln \psi_n - \ln \rho_{k\delta}|^p dv_g \longrightarrow 0, \quad \int_K \left| \frac{\nabla \psi_n}{\psi_n} - \frac{\nabla \rho_{k\delta}}{\rho_{k\delta}} \right|^p dv_g \longrightarrow 0, \quad (\text{as } n \rightarrow +\infty). \quad (3.27)$$

Then we use $\varphi_n := \frac{\varphi}{\psi_n^{p-1-\alpha}}$ as test function in the hypothesis i), obtaining

$$(p-1-\alpha)^2 \int_{\Omega} \frac{|\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \psi_n}{\psi_n^{p-\alpha}} \varphi dv_g \leq (p-1-\alpha) \int_{\Omega} \left(\frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1-\alpha}} \cdot \nabla \varphi \right) dv_g. \quad (3.28)$$

In the case $\alpha < p - 1$ we obtain (3.26) from (3.28) by slight modifications of the proof of Theorem 2.1, so we will omit the proof.

Let $\alpha > p - 1$. We claim that, letting $n \rightarrow +\infty$ in (3.28), and eventually taking a subsequence, we get

$$(p - 1 - \alpha)^2 \int_{\Omega} \frac{|\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \rho_{k\delta}}{\rho_{k\delta}^{p-\alpha}} \varphi \, dv_g \leq (p - 1 - \alpha) \int_{\Omega} \left(\frac{|\nabla \rho|^{p-2} \nabla \rho}{\rho_{k\delta}^{p-1-\alpha}} \cdot \nabla \varphi \right) dv_g. \quad (3.29)$$

In fact, for the right hand side the limit follows by dominated convergence, since

$$\left| \frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1-\alpha}} \cdot \nabla \varphi \right| = |\nabla \rho|^{p-2} |\nabla \rho \cdot \nabla \varphi| \psi_n^{\alpha-p+1} \leq C |\nabla \rho|^{p-1} k^{\alpha-p+1} \in L^1(U).$$

Dealing with the left hand side of (3.28), we set

$$\frac{|\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \psi_n}{\psi_n^{p-\alpha}} = \frac{|\nabla \rho|^{p-2} \nabla \rho}{\psi_n^{p-1}} \cdot \frac{\nabla \psi_n}{\psi_n} \psi_n^{\alpha} =: f_n \cdot g_n.$$

As in the proof of Theorem 2.1, we have

$$f_n \rightarrow \frac{|\nabla \rho|^{p-2} \nabla \rho}{\rho_{k\delta}^{p-1}} \quad \text{in } L^{p'}(U), \quad (3.30)$$

while from the relations

$$|g_n| \leq \left| \frac{\nabla \psi_n}{\psi_n} \right| \cdot k^{\alpha} \rightarrow \left| \frac{\nabla \rho_{k\delta}}{\rho_{k\delta}} \right| \cdot k^{\alpha} \quad \text{in } L^p(U)$$

we obtain that the sequence g_n is bounded in $L^p(U)$. Therefore, up to a subsequence, g_n is weakly convergent in $L^p(U)$. Since

$$g_n \rightarrow \frac{\nabla \rho_{k\delta}}{\rho_{k\delta}} \cdot \rho_{k\delta}^{\alpha} \quad \text{pointwise a. e.,}$$

we have that the convergence is in the weak sense. This fact with (3.30) concludes the claim.

Next step is letting $k \rightarrow +\infty$ in (3.29). Let us rewrite the integrand in the right hand side as

$$\left| \frac{|\nabla \rho|^{p-2} \nabla \rho}{\rho_{k\delta}^{p-1-\alpha}} \cdot \nabla \varphi \right| = \left| |\nabla \rho|^{p-2} \nabla \rho (\rho_{k\delta})^{\frac{\alpha-p}{p'}} (\rho_{k\delta})^{\frac{\alpha-p}{p}+1} \cdot \nabla \varphi \right| \leq C |\nabla \rho|^{p-1} (\rho_{\delta})^{\frac{\alpha-p}{p'}} (\rho_{\delta})^{\frac{\alpha}{p}},$$

which is in $L^1(U)$, since $C |\nabla \rho|^{p-1} (\rho_{\delta})^{\frac{\alpha-p}{p'}} \in L^{p'}(U)$ and $(\rho_{\delta})^{\frac{\alpha}{p}} \in L^p(U)$ by hypothesis ii). Thus we can use the dominated convergence to obtain the limit for the right hand side of (3.29).

In order to pass to the limit for $k \rightarrow +\infty$ in the left hand side of (3.29), we rewrite the integrand as

$$\frac{|\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \rho_{k\delta}}{\rho_{k\delta}^{p-\alpha}} \varphi = \frac{|\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \rho_\delta}{\rho_{k\delta}^{p-\alpha}} \chi_{\{\rho_\delta \leq k\}} \varphi = \frac{|\nabla \rho|^p}{\rho_{k\delta}^{p-\alpha}} \chi_{\{\rho_\delta \leq k\}} \varphi, \quad (3.31)$$

where we have used the fact that $\nabla \rho_\delta = \nabla \rho$. Now, if $\alpha \leq p$ we apply the dominated convergence, since the term in (3.31) is dominated by the function $C \frac{|\nabla \rho|^p}{\delta^{p-\alpha}} \in L^1(U)$. Whereas, if $\alpha > p$ we use the monotone convergence, since $|\nabla \rho|^p \rho_{k\delta}^{\alpha-p} \chi_{\{\rho_\delta \leq k\}} \varphi$ is an increasing sequence of nonnegative functions.

Thus, letting $k \rightarrow +\infty$ in (3.29), we get

$$(p-1-\alpha)^2 \int_\Omega \frac{|\nabla \rho|^p}{\rho_\delta^{p-\alpha}} \varphi \, dv_g \leq (p-1-\alpha) \int_\Omega \left(\frac{|\nabla \rho|^{p-2} \nabla \rho}{\rho_\delta^{p-1-\alpha}} \cdot \nabla \varphi \right) dv_g, \quad (3.32)$$

which is exactly (3.26), since $\nabla \rho_\delta = \nabla \rho$.

As in Theorem 2.1, an application of lemma 2.10 gives

$$\left(\frac{\alpha - p + 1}{p} \right)^p \int_\Omega \rho_\delta^\alpha \frac{|u|^p}{\rho_\delta^p} |\nabla \rho_\delta|^p \, dv_g \leq \int_\Omega \rho_\delta^\alpha |\nabla u|^p \, dv_g. \quad (3.33)$$

Finally, letting $\delta \rightarrow 0$ in (3.33), we conclude the proof. Indeed we can use the dominated convergence for the right hand side, since $\rho_\delta^\alpha |\nabla u|^p \leq C (\rho + \delta)^\alpha \leq C (\rho + 1)^\alpha \in L^1_{loc}(U)$, and apply the Fatou's Lemma for the left hand side. \square

Remark 3.2. If $\alpha \leq p$, then the hypothesis *ii*) in the above Theorem 3.1, can be avoided. Indeed, since $\rho \in W^{1,p}_{loc}(\Omega)$ we get that $\rho^\alpha \in L^1_{loc}(\Omega)$ and from the proof it follows that $\frac{|\nabla \rho|^p}{\rho^{p-\alpha}} \in L^1_{loc}(\Omega)$.

Remark 3.3. Requiring more regularity of the weight ρ , it is possible to drop the hypothesis of the nonnegativity of ρ . In this case, the condition *i*) in Theorem 3.1 should be substituted by the following:

$$-(p-1-\alpha) \rho \Delta_p \rho \geq 0 \quad \text{on } \Omega.$$

As a consequence of Theorem 3.1 (it suffices to take $\alpha = p+q$), we obtain the following Caccioppoli-type inequality for p -subharmonic functions, which is worth of mention:

Corollary 3.4. (*L^p -Caccioppoli-type inequality*) Let $\rho \in L^1_{loc}(M)$ and $q > -1$. Assume that ρ is nonnegative on an open set $\Omega \subset M$, and $\rho \in W^{1,p}_{loc}(\Omega)$, $\rho^q |\nabla \rho|^p, \rho^{p+q} \in L^1_{loc}(\Omega)$. If $\Delta_p \rho \geq 0$ on Ω in weak sense, then we have

$$\left(\frac{q+1}{p} \right)^p \int_\Omega \rho^q |\nabla \rho|^p |u|^p \, dv_g \leq \int_\Omega \rho^{p+q} |\nabla u|^p \, dv_g, \quad u \in C_0^\infty(\Omega). \quad (3.34)$$

Notice that for $q = 0$ and $p = 2$ the above theorem is a version of the classical Caccioppoli inequality on manifolds. See also [PRS] for a version of Caccioppoli inequality related to subharmonic functions on manifolds.

Now we present a possible generalization of Lemma 2.10, and some of its consequences, like the weighted Gagliardo-Nirenberg inequality and the uncertain principle on manifolds.

Lemma 3.5. *Let $h \in L^1_{loc}(\Omega)$ be a vector field and let $A_h \in L^1_{loc}(\Omega)$ be a nonnegative function such that*

- i) $A_h \leq \operatorname{div} h$,
- ii) $\frac{|h|^p}{A_h^{p-1}} \in L^1_{loc}(\Omega)$.

Then for every $u \in C^1_0(\Omega)$, $q \in \mathbb{R}$, $s > 0$ and $a > 1$ we have

$$\int_{\Omega} |u|^s |h|^q dv_g \leq p^{p/a} \left(\int_{\Omega} \frac{|h|^p}{A_h^{p-1}} |\nabla u|^p dv_g \right)^{1/a} \left(\int_{\Omega} \frac{|h|^{qa'}}{A_h^{a'-1}} |u|^{\frac{as-p}{a-1}} dv_g \right)^{1/a'}, \quad (3.35)$$

provided $|h|^q \in L^1_{loc}(\Omega)$.

In particular, setting $w := |h| A_h^{\frac{1-p}{p}}$, we have

1.

$$\left(\int_{\Omega} |u|^s |h|^q dv_g \right)^{1/s} \leq p^{q(p-1)/s} \left(\int_{\Omega} w^p |\nabla u|^p dv_g \right)^{b/p} \left(\int_{\Omega} w^{t\delta} |u|^{\delta} dv_g \right)^{(1-b)/\delta}, \quad (3.36)$$

where $t, \delta > 0$ and

$$\frac{1}{s} = \frac{b}{p} + \frac{1-b}{\delta}, \quad \frac{1}{q} = \frac{1}{p'} + \frac{1}{t\delta}, \quad b = \frac{t(p-1)}{1+t(p-1)}.$$

2.

$$\int_{\Omega} |u|^s dv_g \leq p^{p/a} \left(\int_{\Omega} w^p |\nabla u|^p dv_g \right)^{1/a} \left(\int_{\Omega} \frac{1}{A_h^{a'-1}} |u|^{\frac{as-p}{a-1}} dv_g \right)^{1/a'}, \quad (3.37)$$

where $s > 0$ and $a > 1$.

Proof. By Hölder inequality with exponent a we have

$$\begin{aligned} \int_{\Omega} |u|^s |h|^q dv_g &= \int_{\Omega} |u|^{p/a} A_h^{1/a} |h|^q A_h^{-1/a} |u|^{s-p/a} dv_g \\ &\leq \left(\int_{\Omega} |u|^p A_h dv_g \right)^{1/a} \left(\int_{\Omega} |h|^{qa'} A_h^{-a'/a} |u|^{\frac{as-p}{a-1}} dv_g \right)^{1/a'}, \end{aligned}$$

which by using (2.12), implies (3.35).

From (3.35) we get (3.36) by choosing $a = 1 + \frac{p'}{b\delta}$, and (3.37) by choosing $q = 0$. \square

Specializing h and A_h we obtain from (3.36) and (3.37) a weighted Gagliardo-Nirenberg inequality and an uncertain principle respectively. In particular, choosing h and A_h as in (2.17), we have the following

Theorem 3.6. *Let $\rho \in L^1_{loc}(M)$. Assume that $\rho \in W^{1,p}_{loc}(\Omega)$ is a nonnegative p -superharmonic function on $\Omega \subset M$ and satisfies the hypotheses of Theorem 2.1. Let $\delta > 0$ and $0 \leq b \leq 1$. Then for every $u \in C^1_0(\Omega)$, we have*

$$\left(\int_{\Omega} |u|^s \frac{|\nabla \rho|^{q(p-1)}}{\rho^{q(p-1)}} dv_g \right)^{1/s} \leq \left(\frac{p}{p-1} \right)^{q(p-1)/s} \left(\int_{\Omega} |\nabla u|^p dv_g \right)^{b/p} \left(\int_{\Omega} |u|^{\delta} dv_g \right)^{(1-b)/\delta}, \quad (3.38)$$

where

$$\frac{1}{s} = \frac{b}{p} + \frac{1-b}{\delta}, \quad \frac{1}{q(p-1)} = \frac{1}{p} + \frac{1-b}{b\delta}.$$

In particular, if $\rho = d^{\alpha}$ for some $\alpha \neq 0$ with $|\nabla d| = 1$, then we have

$$\int_{\Omega} \frac{|u|^s}{d^{p-1}} dv_g \leq \left(\frac{p}{|\alpha|(p-1)} \right)^{(p-1)} \left(\int_{\Omega} |\nabla u|^p dv_g \right)^{1/p'} \left(\int_{\Omega} |u|^{\delta} dv_g \right)^{1/p}, \quad (3.39)$$

where $s = p - 1 + \frac{\delta}{p}$.

Notice that for $s = p = 2$ the inequality (3.39) is the weighted Gagliardo-Nirenberg inequality on manifold. Its counterpart in Euclidean setting is largely studied by many authors, see for instance [DV]. Further examples of manifolds and functions ρ satisfying the hypotheses of the above theorem are given in Section 6.

Theorem 3.7. *Let $\rho \in L^1_{loc}(M)$. Assume that $\rho \in W^{1,p}_{loc}(\Omega)$ is a nonnegative p -superharmonic function on $\Omega \subset M$ and satisfies the hypotheses of Theorem 2.1. Let $s > 0$ and $a > 1$. Then for every $u \in C^1_0(\Omega)$, we have*

$$\int_{\Omega} |u|^s dv_g \leq \left(\frac{p}{p-1} \right)^{p/a} \left(\int_{\Omega} |\nabla u|^p dv_g \right)^{1/a} \left(\int_{\Omega} |u|^{\frac{as-p}{a-1}} \frac{\rho^{p(a'-1)}}{|\nabla \rho|^{p(a'-1)}} dv_g \right)^{1/a'}. \quad (3.40)$$

In particular, if $\rho = d^{\alpha}$ for some $\alpha \neq 0$ with $|\nabla d| = 1$, then we have

$$\int_{\Omega} |u|^s dv_g \leq \left(\frac{p}{|\alpha|(p-1)} \right)^{p/a} \left(\int_{\Omega} |\nabla u|^p dv_g \right)^{1/a} \left(\int_{\Omega} |u|^{\frac{as-p}{a-1}} d^{p(a'-1)} dv_g \right)^{1/a'}. \quad (3.41)$$

Notice that if $a = s = p = 2$ the inequality (3.41) in the Euclidean setting coincides with the celebrated uncertain principle with $d = |x|$, the Euclidean norm.

Remark 3.8. Different choices of the vector field h and of the function A_h in Lemma 3.5, produce inequalities different than (3.38)—(3.41). For instance, one can define h and A_h as in (3.25), obtaining a version of (3.38)—(3.41) with further weights.

To end this section, we want to point out that it is possible to extend all the results of this paper considering vector fields of the type $\nabla_\mu u := \mu(\nabla u)$, where μ is a $(1, 1)$ -tensor (say C^1). In this case, replacing ∇ with ∇_μ , a Hardy-type inequality like (2.5) holds provided $\nabla_\mu^* (|\nabla_\mu u|^{p-2} \nabla_\mu u) \geq 0$, where ∇_μ^* stands for the adjoint of ∇_μ . We leave the details to the interested reader. Notice that the study of Hardy inequalities for the vector field ∇_μ was already studied in [Da3], when the support of the manifold is \mathbb{R}^N .

4 Remarks on the best constant

Theorems 2.1 and 3.1 affirm the validity of some Hardy inequalities with an explicit value of the constants involved. In many cases these constants, $\left(\frac{p-1}{p}\right)^p$ and $\left(\frac{|p-1-\alpha|}{p}\right)^p$, result to be sharp. For example in [Da3] the author proves the sharpness of the constant $\left(\frac{|p-1-\alpha|}{p}\right)^p$ involved in the inequality of Theorem 3.1 in several cases. Moreover the question of the existence of functions that realize the best constant is analysed in many papers (for the Euclidean case see for instance [BM, DH, MMP, MS1, MS2]). On the other hand, the knowledge of the best constants for the inequalities plays a crucial role in [AX, CX, Xi].

For the sake of simplicity, we shall focus our attention on the inequality (2.5). We denote by $c(\Omega)$ the best constant in (2.5), namely

$$c(\Omega) := \inf_{u \in D^{1,p}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u|^p dv_g}{\int_\Omega \frac{|u|^p}{\rho^p} |\nabla \rho|^p dv_g}. \quad (4.42)$$

Then, we have the following:

Theorem 4.1. *Under the same hypotheses of Theorem 2.1 we have:*

1. *If $\rho^{\frac{p-1}{p}} \in D^{1,p}(\Omega)$, then $c(\Omega) = \left(\frac{p-1}{p}\right)^p$ and $\rho^{\frac{p-1}{p}}$ is a minimizer.*
2. *If $\rho^{\frac{p-1}{p}} \notin D^{1,p}(\Omega)$, $p \geq 2$ and $c(\Omega) = \left(\frac{p-1}{p}\right)^p$, then the best constant $c(\Omega)$ is not achieved.*

Proof. 1) From (2.5) we have $c(\Omega) \geq \left(\frac{p-1}{p}\right)^p$. Moreover, if $\rho^{\frac{p-1}{p}} \in D^{1,p}(\Omega)$, by computa-

tion

$$\begin{aligned}
\int_{\Omega} \left| \nabla \rho^{\frac{p-1}{p}} \right|^p dv_g &= \int_{\Omega} \left(\frac{p-1}{p} \right)^p (\rho^{-1/p})^p |\nabla \rho|^p dv_g \\
&= \left(\frac{p-1}{p} \right)^p \int_{\Omega} \frac{|\nabla \rho|^p}{\rho} dv_g \\
&= \left(\frac{p-1}{p} \right)^p \int_{\Omega} \frac{\left| \rho^{\frac{p-1}{p}} \right|^p}{\rho^p} |\nabla \rho|^p dv_g.
\end{aligned}$$

Thus, taking $u = \rho^{\frac{p-1}{p}}$, we obtain the infimum in (4.42).

2) Let $u \in C_0^\infty(\Omega)$. We define the functional I as

$$I(u) := \int_{\Omega} |\nabla u|^p dv_g - \left(\frac{p-1}{p} \right)^p \int_{\Omega} \frac{|u|^p}{\rho^p} |\nabla \rho|^p dv_g.$$

We note that the functional I is nonnegative, since (2.5) holds, and the best constant will be achieved if and only if $I(u) = 0$ for some $u \in D^{1,p}(\Omega)$.

Let v be the new variable $v := \frac{u}{\rho^\gamma}$ with $\gamma := \frac{p-1}{p}$. By computation we have

$$\begin{aligned}
|\nabla u|^2 &= |\nabla(v\rho^\gamma)|^2 \\
&= |\gamma|^2 v^2 \rho^{2\gamma-2} |\nabla \rho|^2 + \rho^{2\gamma} |\nabla v|^2 + 2\gamma v \rho^{2\gamma-1} \nabla \rho \cdot \nabla v.
\end{aligned} \tag{4.43}$$

(If ρ is not smooth enough, we can consider ψ_n as in the proof of Theorem 2.1 and after the computation take the limit as $n \rightarrow +\infty$).

We remind that the inequality

$$(\xi - \eta)^s \geq \xi^s - s\eta\xi^{s-1} \tag{4.44}$$

holds for every $\xi, \eta, s \in \mathbb{R}$, with $\xi > 0$, $\xi > \eta$ and $s \geq 1$ (see [GGM]). Applying (4.44) and (4.43), with $s = p/2$, $\xi = |\gamma|^2 v^2 \rho^{2\gamma-2} |\nabla \rho|^2$ and $\eta = -2\gamma v \rho^{2\gamma-1} \nabla \rho \cdot \nabla v - \rho^{2\gamma} |\nabla v|^2$, we have

$$\begin{aligned}
|\nabla u|^p &\geq |\gamma|^p v^p \rho^{p\gamma-p} |\nabla \rho|^p + p |\gamma|^{p-2} \gamma |v|^{p-2} v |\nabla \rho|^{p-2} \nabla \rho \cdot \nabla v \\
&\quad + \frac{p}{2} |\gamma|^{p-2} |v|^{p-2} \rho \nabla \rho^{p-2} |\nabla v|^2.
\end{aligned}$$

Then, taking into account that $v = \frac{u}{\rho^\gamma}$, we have

$$I(u) = \int_{\Omega} |\nabla u|^p dv_g - |\gamma|^p \int_{\Omega} \frac{|u|^p}{\rho^p} |\nabla \rho|^p dv_g \tag{4.45}$$

$$\begin{aligned}
&\geq \int_{\Omega} p |\gamma|^{p-2} \gamma |v|^{p-2} v |\nabla \rho|^{p-2} (\nabla \rho \cdot \nabla v) dv_g \\
&\quad + \int_{\Omega} \frac{p}{2} |\gamma|^{p-2} |v|^{p-2} \rho |\nabla \rho|^{p-2} |\nabla v|^2 dv_g =: I_1(v) + I_2(v).
\end{aligned} \tag{4.46}$$

Re-arranging the expression in $I_1(v)$ and integrating by parts we obtain

$$\begin{aligned} I_1(v) &= \left(\frac{p-1}{p}\right)^{p-1} \int_{\Omega} \nabla(|v|^p) \cdot |\nabla \rho|^{p-2} \nabla \rho \, dv_g \\ &= \left(\frac{p-1}{p}\right)^{p-1} \int_{\Omega} |v|^p (-\Delta_p \rho) \, dv_g \geq 0, \end{aligned}$$

where we have used the hypothesis $-\Delta_p \rho \geq 0$. On the other hand we can rewrite $I_2(v)$ as

$$I_2(v) = \frac{2}{p} |\gamma|^{p-2} \int_{\Omega} \rho |\nabla \rho|^{p-2} \left| \nabla |v|^{p/2} \right|^2 dv_g. \quad (4.47)$$

Thus, we conclude that for every $u \in D^{1,p}(\Omega)$

$$I(u) \geq \frac{2}{p} |\gamma|^{p-2} \int_{\Omega} \rho |\nabla \rho|^{p-2} \left| \nabla |v|^{p/2} \right|^2 dv_g > 0,$$

and this inequality implies the non existence of minimizers in $D^{1,p}(\Omega)$. \square

We end this section by showing a further result that arises from the fact that the best constant, in some cases, is not achieved. Indeed, if the best constant involved in an inequality is not achieved, it is natural to ask if a reminder tem can be added. The next result shows that in the inequality (2.5) one can add an extra term.

Theorem 4.2. *Let $p = 2$ and let ρ be as Theorem 2.1. We define*

$$\Lambda_1 := \inf_{u \in C_0^1(\Omega)} \frac{\int_{\Omega} \rho |\nabla u|^2 \, dv_g}{\int_{\Omega} \rho |u|^2 \, dv_g}.$$

Assume that $\Lambda_1 > 0$. Then

$$\int_{\Omega} |\nabla u|^2 \, dv_g \geq \frac{1}{4} \int_{\Omega} \frac{|u|^2}{\rho^2} |\nabla \rho|^2 \, dv_g + \Lambda_1 \int_{\Omega} u^2 \, dv_g, \quad u \in C_0^1(\Omega). \quad (4.48)$$

Proof. We shall give a sketch of the proof since it is similar to the proof of Theorem 4.1. By using the same notation of the proof of Theorem 4.1, from (4.43), we deduce that

$$I(u) \geq \int_{\Omega} \rho |\nabla v|^2 \, dv_g \geq \Lambda_1 \int_{\Omega} \rho |v|^2 \, dv_g = \Lambda_1 \int_{\Omega} u^2 \, dv_g,$$

where we have used the fact that $-\Delta \rho \geq 0$, the definition of Λ_1 and $v = u/\rho^{1/2}$. This concludes the proof. \square

An example of manifold where the hypothesis that $\Lambda_1 > 0$ is satisfied is the following. Let ρ be a nonnegative superharmonic function on \mathbb{R}^N and let $\Omega \subset \mathbb{R}^N$ a bounded open

set. Then ρ belongs to the Muckenhoupt class A_1 and this implies that $\Lambda_1 > 0$ (indeed it suffices to combine Theorems 3.59 and 15.21 of [HKM]). In particular, with the choice $\rho := |x|^{2-N}$, $N > 2$ and $\Omega \subset \mathbb{R}^N$ a bounded open set, (4.48) reads as

$$\int_{\Omega} |\nabla u|^2 dx \geq \frac{(N-2)^2}{4} \int_{\Omega} \frac{|u|^2}{|x|^2} dx + \Lambda_1 \int_{\Omega} u^2 dx, \quad u \in C_0^1(\Omega),$$

which is the celebrated inequality proved in [BV]. See also [Da2, GGM] for related results in Euclidean and subelliptic setting for $p > 1$ and for further references.

5 First order interpolation inequalities

In this section we shall study some inequalities of Hardy-Sobolev type. As already said above, interpolation inequalities as well as the knowledge of an estimate of the best constant have an important role in several areas of mathematical science. Thus we shall address some efforts to keep track of explicit values of the involved constants.

We shall assume that the Sobolev inequality

$$S(p) \left(\int_{\Omega} |u|^{p^*} dv_g \right)^{1/p^*} \leq \left(\int_{\Omega} |\nabla u|^p dv_g \right)^{1/p}, \quad u \in C_0^\infty(\Omega) \quad (S)$$

holds for some $p^* > 0$, and the Hardy inequality

$$H(\alpha, p) \int_{\Omega} \rho^\alpha \frac{|u|^p}{\rho^p} |\nabla \rho|^p dv_g \leq \int_{\Omega} \rho^\alpha |\nabla u|^p dv_g, \quad u \in C_0^\infty(\Omega) \quad (H_\alpha)$$

holds for an exponent $\alpha \in \mathbb{R}$.

In some cases, the validity of (S) implies that (H_α) holds as well. Indeed, let $N > 2$ and let M be a N -dimensional complete and connected Riemannian manifold with infinite volume, if (S) holds with $p = 2$ and $p^* = 2N/(N-2)$ then M is hyperbolic (see [Ca1]). In this case, from Theorem 2.6 we have that a Hardy inequality holds. Therefore, there exists a nonnegative nonconstant superharmonic function ρ and hence (H_α) holds with $p = 2$ and $\alpha < 1$ (see Theorem 3.1 and Remark 3.2).

In order to state our main result of this section, we need the following preliminary theorem:

Theorem 5.1. *Assume that (S) holds on Ω . Let $\theta \in \mathbb{R}$ and $\rho \geq 0$ be a function such that (H_α) holds with $\alpha = p\theta$. Then there exists $C_2 > 0$ such that*

$$C_2 \left(\int_{\Omega} \rho^{p^*\theta} |u|^{p^*} dv_g \right)^{1/p^*} \leq \left(\int_{\Omega} \rho^{p\theta} |\nabla u|^p dv_g \right)^{1/p}, \quad u \in C_0^\infty(\Omega). \quad (5.49)$$

Moreover

$$C_2 = S(p) \frac{H(p\theta, p)^{1/p}}{|\theta| + H(p\theta, p)^{1/p}}.$$

In particular, if $\rho \in L^1_{loc}(\Omega)$ is a nonnegative function satisfying the hypotheses of Theorem 3.1 with $\alpha = p\theta$ and (S) holds, then we obtain (5.49) with

$$C_2 = S(p) \frac{|p-1-p\theta|}{p|\theta| + |p-1-p\theta|}$$

Proof. The case $\theta = 0$ corresponds to the Sobolev inequality. Let $\theta \neq 0$. Let $u \in C_0^\infty(\Omega)$ and define v as $v := \rho^\theta u$. By computation we have

$$\begin{aligned} |\nabla v|^p &= |\nabla(\rho^\theta u)|^p = |\rho^\theta \nabla u + \theta \rho^{\theta-1} u \nabla \rho|^p \\ &\leq (\rho^\theta |\nabla u| + |\theta| \rho^{\theta-1} |u| |\nabla \rho|)^p \\ &= \left(\rho^\theta |\nabla u| + \frac{|\theta|}{H^{1/p}} H^{1/p} \rho^{\theta-1} |u| |\nabla \rho| \right)^p \end{aligned} \quad (5.50)$$

where, for sake of brevity, $H = H(p\theta, p)$ and $S = S(p)$. By using the inequality

$$(a + \frac{1-\epsilon}{\epsilon} b)^p \leq \epsilon^{1-p} a^p + \frac{1-\epsilon}{\epsilon^p} b^p \quad (0 < \epsilon < 1, \quad a, b > 0)$$

with $\epsilon := \frac{H^{1/p}}{H^{1/p} + |\theta|}$, $a := \rho^\theta |\nabla u|$ and $b := H^{1/p} \rho^{\theta-1} |u| |\nabla \rho|$, we have

$$|\nabla v|^p \leq \epsilon^{1-p} \rho^{p\theta} |\nabla u|^p + \frac{1-\epsilon}{\epsilon^p} H \rho^{p\theta-p} |u|^p |\nabla \rho|^p. \quad (5.51)$$

Then, by (S) and using (H_α) with $\alpha = p\theta$, we obtain

$$\begin{aligned} \left(\int_\Omega \rho^{p^*\theta} |u|^{p^*} dv_g \right)^{p/p^*} &= \left(\int_\Omega |v|^{p^*} dv_g \right)^{p/p^*} \leq S^{-p} \int_\Omega |\nabla v|^p \\ &\leq S^{-p} \left[\epsilon^{1-p} \int_\Omega \rho^{p\theta} |\nabla u|^p dv_g + \frac{1-\epsilon}{\epsilon^p} H \int_\Omega \rho^{p\theta} \frac{|u|^p}{\rho^p} |\nabla \rho|^p dv_g \right] \\ &\leq S^{-p} \left[\epsilon^{1-p} + \frac{1-\epsilon}{\epsilon^p} \right] \int_\Omega \rho^{p\theta} |\nabla u|^p dv_g = (S\epsilon)^{-p} \int_\Omega \rho^{p\theta} |\nabla u|^p dv_g, \end{aligned} \quad (5.52)$$

which concludes the proof. \square

Theorem 5.2. Assume that (S) holds on Ω with $p^* > p$. Let $\theta \in \mathbb{R}$ and $\rho \geq 0$ be a function such that (H_α) holds with $\alpha = p\theta$. Let $r > 0$, $0 \leq a \leq 1$, γ, ϵ, σ and δ be real numbers satisfying the following relations

$$\frac{1}{p} \geq \frac{1}{r} \geq \frac{1-a}{p} + \frac{a}{p^*}, \quad (5.53)$$

$$\gamma + \frac{p^*(r-p)}{r(p^*-p)} = (1-\theta)a + \delta(1-a) \quad (5.54)$$

and

$$\epsilon = \theta a + \sigma(1-a). \quad (5.55)$$

Then there exists $C_3 > 0$ such that

$$C_3 \left(\int_{\Omega} \frac{|u|^r}{\rho^{\gamma r}} |\nabla \rho|^{(\gamma+\epsilon)r} dv_g \right)^{1/r} \leq \left(\int_{\Omega} \rho^{\theta p} |\nabla u|^p dv_g \right)^{a/p} \left(\int_{\Omega} |u|^p \frac{|\nabla \rho|^{(\delta+\sigma)p}}{\rho^{\delta p}} dv_g \right)^{(1-a)/p}, \quad (5.56)$$

$u \in C_0^\infty(\Omega)$

that is

$$C_3 \left| u \frac{|\nabla \rho|^{\gamma+\epsilon}}{\rho^\gamma} \right|_{L^r} \leq |\rho^\theta |\nabla u||_{L^p}^a \left| u \frac{|\nabla \rho|^{\delta+\sigma}}{\rho^\delta} \right|_{L^p}^{1-a}, \quad (5.57)$$

provided $\frac{|\nabla \rho|^{p\sigma}}{\rho^{p\delta}} \in L_{loc}^1(\Omega)$.

Moreover

$$C_3 = C_2^{\frac{p^*(r-p)}{r(p^*-p)}} H(p\theta, p)^{\frac{a}{p} - \frac{p^*(r-p)}{pr(p^*-p)}}.$$

In particular, if $\rho \in L_{loc}^1(\Omega)$ is a nonnegative function satisfying the hypotheses of Theorem 3.1 with $\alpha = p\theta$ and (S) holds, then we obtain (5.56) with

$$C_3 = C_2^{\frac{p^*(r-p)}{r(p^*-p)}} \left(\frac{|p-1-p\theta|}{p} \right)^{a - \frac{p^*(r-p)}{r(p^*-p)}}.$$

Proof. From condition (5.53) it follows that $p^* \geq r \geq p$. We shall distinguish three cases.

Case: $r = p^*$. From (5.53) necessarily we have $a = 1$ and hence from (5.54) and (5.55) $\epsilon = \theta = -\gamma$. The inequality to prove is actually the thesis of Theorem 5.1.

Case: $r = p$. If $a = 0$ there is nothing to prove. If $a = 1$, then the thesis is the inequality (H_α) . Let $0 < a < 1$. By using (5.54) and (5.55) we have

$$\int_{\Omega} \frac{|u|^r}{\rho^{\gamma r}} |\nabla \rho|^{(\gamma+\epsilon)r} dv_g = \int_{\Omega} \frac{|u|^{ap}}{\rho^{(1-\theta)ap}} |\nabla \rho|^{ap} \frac{|u|^{(1-a)p}}{\rho^{(1-a)\delta p}} |\nabla \rho|^{(\delta+\sigma)p} dv_g.$$

Now the claim follows applying Hölder inequality with exponent $1/a$ and then Hardy inequality (H_α) .

Case: $p^* > r > p$. Let $q \in \mathbb{R}$ be a parameter that we shall fix later. Using Hölder inequality with exponent $s > 1$ we obtain

$$\begin{aligned} \int_{\Omega} \frac{|u|^r}{\rho^{\gamma r}} |\nabla \rho|^{(\gamma+\epsilon)r} dv_g &= \int_{\Omega} |u|^{r-q} \rho^{p^*\theta/s} \frac{|u|^q}{\rho^{\gamma r + p^*\theta/s}} |\nabla \rho|^{(\gamma+\epsilon)r} dv_g \\ &\leq \left(\int_{\Omega} |u|^{(r-q)s} \rho^{p^*\theta} dv_g \right)^{1/s} \left(\int_{\Omega} \frac{|u|^{qs'}}{\rho^{(\gamma r + p^*\theta/s)s'}} |\nabla \rho|^{(\gamma+\epsilon)rs'} dv_g \right)^{1/s'}. \end{aligned} \quad (5.58)$$

Now we apply Hölder inequality with exponent $t > 1$ to the second term of (5.58) and obtain

$$\begin{aligned} \int_{\Omega} \frac{|u|^{qs'}}{\rho^{(\gamma r + p^* \theta/s)s'}} |\nabla \rho|^{(\gamma + \epsilon)rs'} dv_g &= \int_{\Omega} \frac{|u|^{qs'/t}}{\rho^{qs'/t}} |\nabla \rho|^{qs'/t} \rho^{p\theta/t} \frac{|u|^{qs'/t'} |\nabla \rho|^{(\gamma + \epsilon)rs' - qs'/t}}{\rho^{(\gamma r + p^* \theta/s)s' - qs'/t + p\theta/t}} dv_g \\ &\leq \left(\int_{\Omega} \frac{|u|^{qs'}}{\rho^{qs'}} |\nabla \rho|^{qs'} \rho^{p\theta} dv_g \right)^{1/t} \left(\int_{\Omega} \frac{|u|^{qs'} |\nabla \rho|^{(\gamma + \epsilon)rs't' - qs't'/t}}{\rho^{(\gamma r + p^* \theta/s)s't' - qs't'/t + p\theta t'/t}} dv_g \right)^{1/t'} \end{aligned} \quad (5.59)$$

Now, requiring that the following conditions are satisfied

$$qs' = p, \quad (r - q)s = p^*, \quad (5.60)$$

we get

$$s = \frac{p^* - p}{r - p} > 1, \quad (5.61)$$

since (5.53) holds. Using (5.60), by (5.58) and (5.59) we have

$$\begin{aligned} \int_{\Omega} \frac{|u|^r}{\rho^{\gamma r}} |\nabla \rho|^{(\gamma + \epsilon)r} dv_g &\leq \left(\int_{\Omega} |u|^{p^*} \rho^{p^* \theta} dv_g \right)^{1/s} \\ &\left(\int_{\Omega} \frac{|u|^p}{\rho^p} |\nabla \rho|^p \rho^{p\theta} dv_g \right)^{1/s't} \left(\int_{\Omega} \frac{|u|^p |\nabla \rho|^{(\gamma + \epsilon)rs't' - pt'/t}}{\rho^{(\gamma r + p^* \theta/s)s't' - pt'/t + p\theta t'/t}} dv_g \right)^{1/s't'} \\ &\leq C_2^{-p^*/s} H(p\theta, p)^{-1/s't} \left(\int_{\Omega} \rho^{p\theta} |\nabla u|^p dv_g \right)^{p^*/ps + 1/s't} \\ &\left(\int_{\Omega} \frac{|u|^p |\nabla \rho|^{(\gamma + \epsilon)rs't' - pt'/t}}{\rho^{(\gamma r + p^* \theta/s)s't' - pt'/t + p\theta t'/t}} dv_g \right)^{1/s't'}, \end{aligned} \quad (5.62)$$

where, in the last inequality, we have used (5.49) and (H_{α}) with $\alpha = p\theta$. To conclude we have to choose $t > 1$ such that

$$(\gamma + \epsilon)rs't' - pt'/t = p(\delta + \sigma), \quad (\gamma r + p^* \theta/s)s't' - pt'/t + p\theta t'/t = p\delta, \quad (5.63)$$

and

$$p^*/ps + 1/s't = ar/p, \quad 1/s't' = (1 - a)r/p. \quad (5.64)$$

First of all, note that in (5.59) we can make the choice $t = \frac{sp}{s'(asr - p^*)} > 1$, since (5.53) holds. Using the expressions of t and s , equalities (5.64) follow by simple computations. Moreover, from (5.64) we obtain that $s't' = \frac{p}{r(1-a)}$ and $\frac{t'}{t} = \frac{asr - p^*}{sr(1-a)}$. Using this two expressions and the conditions (5.54) and (5.55) we get also (5.63). This concludes the proof. \square

Remark 5.3. The condition (5.55) takes into account the presence of the $|\nabla\rho|$ in the weights appearing in (5.56) and it is also a necessary condition. Indeed, to see the necessity of (5.55) we argue as follows. Assume that Theorem 5.2 were true. If (S) and (H_α) hold with a function ρ , then those inequalities still hold with the function $\lambda\rho$ for every $\lambda > 0$, and hence the conclusion of Theorem 5.2 holds replacing ρ with $\lambda\rho$. By homogeneous consideration one derives the necessity of (5.55).

Remark 5.4. Since the condition (5.55) is a requirement on the parameters ϵ and σ , if $|\nabla\rho| = 1$, these parameters do not appear in the inequality (5.56). Therefore, condition (5.55) is always fulfilled (i.e. choosing $\epsilon = a\theta$ and $\sigma = 0$). The next corollary deals with a generalization of this case.

Corollary 5.5. *Assume that (S) holds on Ω with $p^* > p$. Let $\theta \in \mathbb{R}$ and $\rho \geq 0$ be a function such that (H_α) holds with $\alpha = p\theta$ and $\rho = d^\beta$ with $\beta \in \mathbb{R}$ and $|\nabla d| = 1$. Let $r > 0$, $0 \leq a \leq 1$ and γ, δ be real numbers satisfying (5.53) and*

$$\gamma + \frac{p^*(r-p)}{r(p^*-p)} = (1-\beta\theta)a + \delta(1-a). \quad (5.65)$$

Then there exists $C'_3 > 0$ such that

$$C'_3 \left(\int_{\Omega} \frac{|u|^r}{d^{\gamma r}} dv_g \right)^{1/r} \leq \left(\int_{\Omega} d^{\beta\theta p} |\nabla u|^p dv_g \right)^{a/p} \left(\int_{\Omega} \frac{|u|^p}{d^{\delta p}} dv_g \right)^{(1-a)/p}, \quad u \in C_0^\infty(\Omega) \quad (5.66)$$

that is

$$C'_3 \left| \frac{u}{\rho^\gamma} \right|_{L^r} \leq |\rho^{\beta\theta} |\nabla u||_{L^p}^a \left| \frac{u}{\rho^\delta} \right|_{L^p}^{1-a}, \quad (5.67)$$

provided $\frac{1}{\rho^{\beta\delta}} \in L_{loc}^1(\Omega)$. In particular, $C'_3 = C_3 \beta^{\gamma+\beta\theta a-(1-a)\delta}$.

Remark 5.6. Notice that, if in the previous corollary we take $p^* = \frac{pN}{N-p}$, condition (5.65) becomes

$$\frac{1}{r} - \frac{\gamma}{N} = \frac{1}{p} + \frac{a}{N} (\beta\theta - 1) - \frac{\delta}{N} (1-a), \quad (5.68)$$

and (5.66) is a particular case on manifold of a result obtained by Caffarelli, Kohn and Nirenberg in [CKN] in Euclidean setting.

6 Some Applications

In what follows we apply the results proved in the above sections to concrete cases, obtaining Hardy inequalities which in some cases are new. For sake of brevity we shall limit ourselves to show some applications of Theorems 2.1 and 3.1 by specializing the

function ρ . With the same technique it is possible to obtain applications of the other theorems presented in the previous sections (Caccioppoli inequality, uncertain principle, Gagliardo-Nirenberg inequality, first order interpolation inequalities, and so on). We leave the details to the interested reader.

6.1 Hardy inequality involving the distance from the boundary

In order to prove a Hardy inequality involving the distance from the boundary, we need the following result, which is an immediate consequence of Theorems 2.1 and 3.1.

Theorem 6.1. *Let \overline{M} be a compact Riemannian manifold with boundary of class C^∞ , let φ_1 be the first eigenfunction related to the first eigenvalue of the p -Laplacian⁴, and $\alpha < p - 1$. If $\varphi_1 > 0$, then the following inequality holds on M*

$$\left(\frac{p-1-\alpha}{p}\right)^p \int_M \varphi_1^\alpha \frac{|u|^p}{\varphi_1^p} |\nabla \varphi_1|^p dv_g \leq \int_M \varphi_1^\alpha |\nabla u|^p dv_g, \quad u \in C_0^\infty(M). \quad (6.70)$$

In particular we have

$$\left(\frac{p-1}{p}\right)^p \int_M \frac{|u|^p}{\varphi_1^p} |\nabla \varphi_1|^p dv_g \leq \int_M |\nabla u|^p dv_g, \quad u \in C_0^\infty(M). \quad (6.71)$$

The main theorem of this section is the following:

Theorem 6.2. *Let \overline{M} be a compact Riemannian manifold with boundary of class C^∞ , let φ_1 be the first eigenfunction related to the first eigenvalue of the p -Laplacian. Assume that $\varphi_1 \in C^1(\overline{M})$, $\varphi_1 > 0$ on M and $|\nabla \varphi_1| \neq 0$ on ∂M .*

Denoted by $d(x) := \text{dist}(x, \partial M)$, there exists a constant $c > 0$ such that

$$c \int_M \frac{|u|^p}{d^p} dv_g \leq \int_M |\nabla u|^p dv_g, \quad u \in C_0^\infty(M). \quad (6.72)$$

The proof of the above theorem relies on the following result, which is worth of mention:

Theorem 6.3. *Let \overline{M} be a compact Riemannian manifold with boundary of class C^∞ , let $\varphi_1 > 0$ be the first eigenfunction related to the first eigenvalue λ_1 of the p -Laplacian, and $0 < s < p - 1$. Then the following inequality holds*

$$\lambda_1 \frac{(p-1-s)^{(p-1)}}{p^p} \int_M \varphi_1^s |u|^p dv_g \leq \int_M \varphi_1^s |\nabla u|^p dv_g, \quad u \in C_0^\infty(M). \quad (6.73)$$

⁴This means that (λ_1, φ_1) is a solution of the problem

$$\begin{cases} -\Delta_p \varphi = \lambda |\varphi|^{p-2} \varphi & \text{on } M, \\ \varphi = 0 & \text{on } \partial M, \end{cases} \quad (6.69)$$

and $\lambda_1 := \min\{\lambda : (\lambda, \varphi) \text{ solves (6.69)}\}$.

Proof. Set $\phi := \varphi_1^{s/(p-1)}$. By computation we have

$$-\Delta_p \phi = \frac{|\nabla \phi|^p}{\phi} (p-1-s) \left(\frac{p-1}{s} \right) + \lambda_1 \left(\frac{s}{p-1} \right)^{p-1} \phi^{p-1}.$$

Choosing $h := -|\nabla \phi|^{p-2} \nabla \phi$ and $A_h := -\Delta_p \phi$, an application of Lemma 2.10 yields

$$\begin{aligned} \lambda_1 \left(\frac{s}{p-1} \right)^{p-1} \int_M \phi^{p-1} |u|^p dv_g &\leq \int_M (-\Delta_p \phi) |u|^p dv_g \leq p^p \int_M \frac{|\nabla \phi|^{p(p-1)}}{(-\Delta_p \phi)^{p-1}} |\nabla u|^p dv_g \\ &\leq p^p \left(\frac{s}{p-1} \right)^{p-1} \left(\frac{1}{p-1-s} \right)^{p-1} \int_M \phi^{p-1} |\nabla u|^p dv_g. \end{aligned}$$

This last chain of inequalities concludes the proof. \square

Proof of Theorem 6.2. For a fixed number $\gamma > 0$, we denote by Ω^γ and Ω_γ respectively the sets $\Omega^\gamma := \{x : d(x) < \gamma\}$ and $\Omega_\gamma := \{x : d(x) \geq \gamma\}$.

Let φ_1 be such that $\|\varphi_1\|_\infty \leq 1$. By continuity argument, we have that there exist $\epsilon, b > 0$ such that

$$|\nabla \varphi_1(x)| \geq b > 0, \quad \text{for } x \in \Omega^\epsilon.$$

Since φ_1 is a Lipschitz continuous function, we obtain that there exist $L > 0$ such that

$$\varphi_1(x) \leq Ld(x), \quad x \in M.$$

We set

$$l_\epsilon := \min_{\Omega_\epsilon} \varphi_1.$$

From (6.71) and (6.73) we get respectively

$$\begin{aligned} \int_M |\nabla u|^p dv_g &\geq \left(\frac{p-1}{p} \right)^p \int_{\Omega^\epsilon} \frac{|u|^p}{\varphi_1^p} |\nabla \varphi_1|^p dv_g \geq \left(\frac{p-1}{p} \right)^p \frac{b^p}{L^p} \int_{\Omega^\epsilon} \frac{|u|^p}{d^p} dv_g, \\ \int_M |\nabla u|^p dv_g &\geq \lambda_1 \frac{(p-1-s)^{(p-1)}}{p^p} \int_{\Omega_\epsilon} \varphi_1^s |u|^p dv_g \geq \lambda_1 \frac{(p-1-s)^{(p-1)}}{p^p} l_\epsilon^s \epsilon^p \int_{\Omega_\epsilon} \frac{|u|^p}{d^p} dv_g. \end{aligned}$$

Choosing $2c := \min \left\{ \left(\frac{p-1}{p} \right)^p \frac{b^p}{L^p}, \lambda_1 \frac{(p-1-s)^{(p-1)}}{p^p} l_\epsilon^s \epsilon^p \right\}$ and summing up the above estimates we obtain the claim. \square

6.2 Hardy inequality for p -hyperbolic manifold

In this section we establish Hardy inequalities involving the Green function of the operator $-\Delta_p$. The case $p = 2$ is already proved in [LW]. We have the following:

Theorem 6.4. *Let (M, g) be a p -hyperbolic manifold, let G_x be the Green function for Δ_p with pole at x , and $\alpha \in \mathbb{R}$. Then the following inequality holds*

$$\left(\frac{|p-1-\alpha|}{p}\right)^p \int_{M \setminus \{x\}} G_x^\alpha \frac{|\nabla G_x|^p}{G_x^p} |u|^p dv_g \leq \int_{M \setminus \{x\}} G_x^\alpha |\nabla u|^p dv_g, \quad u \in C_0^\infty(M \setminus \{x\}). \quad (6.74)$$

In particular, we have

$$\left(\frac{p-1}{p}\right)^p \int_{M \setminus \{x\}} \frac{|\nabla G_x|^p}{G_x^p} |u|^p dv_g \leq \int_{M \setminus \{x\}} |\nabla u|^p dv_g, \quad u \in C_0^\infty(M \setminus \{x\}). \quad (6.75)$$

and, if $p < N$,

$$\left(\frac{p-1}{p}\right)^p \int_M \frac{|\nabla G_x|^p}{G_x^p} |u|^p dv_g \leq \int_M |\nabla u|^p dv_g, \quad u \in C_0^\infty(M). \quad (6.76)$$

Proof. We know that $G_x \in W_{loc}^{1,p}(M)$ is a nonnegative function on M . Moreover the hypotheses of Theorem 3.1 are fulfilled; in fact

- i) $-\Delta_p G_x = 0$ in $M \setminus \{x\}$
- ii) $\frac{|\nabla G_x|^p}{G_x^{p-\alpha}}, G_x^\alpha \in L_{loc}^1(M \setminus \{x\})$.

Then, by Theorem 3.1, inequality (6.74) holds. In particular, taking $\alpha = 0$, we obtain the inequality (6.75). Moreover, if $p < N$, we are in the position to apply Corollary 2.3, because $\{x\}$ is a set of zero p -capacity (see Theorem 2.27 in [HKM]), and then we can use Proposition A-1; this proves that also (6.76) holds. \square

6.3 Hardy inequality on Cartan-Hadamard manifold

In what follows (M, g) will denote a Cartan-Hadamard manifold, that is, a connected, simply connected, complete Riemannian manifold of dimension $N \geq 2$, of nonpositive sectional curvature (see [Ca2, Ho1, LW] for further details). Let $o \in M$ be a fixed point and denote by r the distance function from o . We have the following:

Theorem 6.5. *Let (M, g) be a Cartan-Hadamard manifold and $\alpha \in \mathbb{R}$.*

If $(N-p)(p-1-\alpha) > 0$, we have

$$\left(\frac{(N-p)(p-1-\alpha)}{p(p-1)}\right)^p \int_{M \setminus \{o\}} r^{\alpha \frac{p-N}{p-1}} \frac{|u|^p}{r^p} dv_g \leq \int_{M \setminus \{o\}} r^{\alpha \frac{p-N}{p-1}} |\nabla u|^p dv_g, \quad u \in C_0^\infty(M \setminus \{o\}), \quad (6.77)$$

and, in particular, if $1 < p < N$,

$$\left(\frac{N-p}{p}\right)^p \int_M \frac{|u|^p}{r^p} dv_g \leq \int_M |\nabla u|^p dv_g, \quad u \in C_0^\infty(M). \quad (6.78)$$

If $p < N$, setting $\Omega := r^{-1}(]0, 1[)$ or $\Omega := r^{-1}(]1, +\infty[)$, we have

$$\left(\frac{p-1}{p}\right)^p \int_{\Omega} \frac{|u|^p}{|r \ln r|^p} dv_g \leq \int_{\Omega} |\nabla u|^p dv_g, \quad u \in C_0^\infty(\Omega). \quad (6.79)$$

Proof. In $M \setminus \{o\}$ we define $\rho = r^\beta$ with $\beta \in \mathbb{R}$ that will be chosen later. The function $\rho \in W_{loc}^{1,p}(M \setminus \{o\})$ is nonnegative on $M \setminus \{o\}$.

By computation

$$\begin{aligned} \Delta_p \rho &= \operatorname{div} (|\nabla \rho|^{p-2} \nabla \rho) \\ &= \operatorname{div} (|\beta|^{p-2} r^{(\beta-1)(p-2)} \beta r^{\beta-1} \nabla r) \\ &= |\beta|^{p-2} \beta \operatorname{div} (r^{(\beta-1)(p-1)} \nabla r) \\ &= |\beta|^{p-2} \beta [(\beta-1)(p-1) r^{(\beta-1)(p-1)-1} + r^{(\beta-1)(p-1)} \Delta r] \\ &= |\beta|^{p-2} \beta r^{(\beta-1)(p-1)-1} [(\beta-1)(p-1) + r \Delta r], \end{aligned}$$

then we have

$$-(p-1-\alpha) \Delta_p \rho = -(p-1-\alpha) |\beta|^{p-2} \beta r^{(\beta-1)(p-1)-1} [(\beta-1)(p-1) + r \Delta r]. \quad (6.80)$$

The function r satisfies the relations

$$|\nabla r| = 1, \quad \Delta r \geq \frac{N-1}{r}, \quad (6.81)$$

(see [Ca2]). Then choosing $\beta = \frac{p-N}{p-1}$, we have that $(\beta-1)(p-1) + r \Delta r \geq 0$, and $-(p-1-\alpha) \Delta_p \rho \geq 0$, that is the hypothesis i) of Theorem 3.1 is fulfilled. Since $\frac{|\nabla \rho|^p}{\rho^{p-\alpha}} = |\beta|^p \frac{1}{r^{p-\alpha}} \in L_{loc}^1(M \setminus \{o\})$ and $\rho^\alpha = r^{\beta\alpha} \in L_{loc}^1(M \setminus \{o\})$, we are in a position to apply Theorem 3.1, obtaining the inequality (6.77).

In particular, taking $\alpha = 0$ and using Corollary 2.3 we get (6.78). In fact, since $p < N$, $\{o\}$ is a set of zero p -capacity (see Theorem 2.27 in [HKM]), and then we can use Proposition A-1.

If $\Omega = r^{-1}(]0, 1[)$, then from Theorem 2.1 with the choice $\rho := -\ln r$ we have that inequality (6.79) holds for every $\rho \in C_0^\infty(\Omega \setminus \{o\})$. However in this case Ω is p -hyperbolic, and $\{o\}$ is a set of zero p -capacity. Thus applying Corollary 2.3 we obtain (6.79).

Finally, if $\Omega = r^{-1}(]1, +\infty[)$, inequality (6.79) follows from Theorem 2.1 with the choice $\rho := \ln r$. \square

6.4 Hardy inequalities involving the distance from the soul of a manifold

Let (M, g) be a complete noncompact Riemannian manifold, of dimension $N \geq 2$, with nonnegative sectional curvatures. A result due to Cheeger and Gromoll asserts that there

exists a compact embedded totally convex submanifold S with empty boundary, whose normal bundle is diffeomorphic to M (see [CG]). The submanifold S , called "soul" of M , is not necessarily unique but every two souls of M are isometric. "*Totally convex*" means that any geodesic arc in M connecting two points in S (which may coincide) lies entirely in S . In particular, S is connected, totally geodesic in M , and has nonnegative sectional curvature. Moreover $0 \leq \dim S < \dim M$.

Denote by $r : M \setminus S \rightarrow \mathbb{R}$ the distance function to S . We have that r is smooth on $M \setminus S$ and $|\nabla r| = 1$ on $M \setminus S$. Now we suppose that radial sectional curvature K_r , that is sectional curvature of two-planes containing the direction ∇r , satisfies

$$0 \leq K_r \leq \frac{c_N(1 - c_N)}{r^2}, \quad (6.82)$$

where $c_N = \frac{N-2}{N}$; then we have

$$\Delta r \geq \frac{c_N(N - s - 1)}{r}, \quad (6.83)$$

where $s = \dim S$ (see [EF]). We have the following:

Theorem 6.6. *Let (M, g) be a Riemannian manifold with nonnegative curvature. Suppose that (6.82) is fulfilled.*

Let $G := c_N(N - s - 1) - p + 1$. If $G \cdot (p - 1 - \alpha) > 0$, we have

$$\left(\frac{G \cdot (p - 1 - \alpha)}{p(p - 1)} \right)^p \int_{M \setminus S} r^{-\alpha \frac{G}{p-1}} \frac{|u|^p}{r^p} dv_g \leq \int_{M \setminus S} r^{-\alpha \frac{G}{p-1}} |\nabla u|^p dv_g, \quad u \in C_0^\infty(M \setminus S) \quad (6.84)$$

Moreover, if $G > 0$, we have

$$\left(\frac{G}{p} \right)^p \int_M \frac{|u|^p}{r^p} dv_g \leq \int_M |\nabla u|^p dv_g, \quad u \in C_0^\infty(M). \quad (6.85)$$

Proof. Let $\rho = r^\beta$ in $M \setminus S$ with $\beta = -\frac{G}{p-1}$. Arguing as in the proof of Theorem 6.5, using (6.80) and (6.83), we obtain $-(p - 1 - \alpha) \Delta_p \rho \geq 0$. Since $\rho \in W_{loc}^{1,p}(M \setminus S)$, $\frac{|\nabla \rho|^p}{\rho^{p-\alpha}} = |\beta|^p \frac{1}{r^{p-\alpha}} \in L_{loc}^1(M \setminus S)$ and $\rho^\alpha = r^{\beta\alpha} \in L_{loc}^1(M \setminus S)$, the hypotheses of Theorem 3.1 are fulfilled, and (6.84) follows.

In particular, if $\alpha = 0$, (6.84) becomes

$$\left(\frac{G}{p} \right)^p \int_{M \setminus S} \frac{|u|^p}{r^p} dv_g \leq \int_{M \setminus S} |\nabla u|^p dv_g, \quad u \in C_0^\infty(M \setminus S).$$

Now, the hypothesis $G > 0$ implies that $N - p > s$. In fact, by the fact that $G = c_N(N - s - 1) - p + 1 = \frac{N-2}{N}(N - s - 1) - p + 1 > 0$, by simple computations we get

$$\begin{aligned} (N - 2)(N - s - 1) - N(p + 1) &= (N - 2)(N - s) - Np \\ &= N(N - s) - 2(N - s) - Np \\ &= N(N - s - p) - 2(N - s) > 0, \end{aligned}$$

which implies $N - s - p > 2\frac{N-s}{N} > 0$. Then S is a set of zero p -capacity (see Theorem 2.27 in [HKM]), and we can use Proposition A-1 and Corollary 2.3 to obtain inequality (6.85). \square

6.5 Hyperbolic plane

Let \mathbb{H} be the upper half-plane $\mathbb{C}_+ = \{z = x + iy : \text{Im}z = y > 0\}$ equipped with the Poincaré metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. This space is a Riemannian manifold that models the *two dimensional hyperbolic space*. We will see that, using Theorem 2.1, one can deduce the Hardy inequality for the hyperbolic plane (see for instance [He] and references therein).

Theorem 6.7. *Let $\alpha \in \mathbb{R}$. Then, for every $u \in C_0^\infty(\mathbb{C}_+)$, we have the following:*

$$\left(\frac{|p-1-\alpha|}{p}\right)^p \int_{\mathbb{C}_+} |u|^p \frac{dA(z)}{(\text{Im}z)^{p-\alpha}} \leq 2^{p/2} \int_{\mathbb{C}_+} (\text{Im}z)^\alpha \left(|\partial u(z)|^2 + |\bar{\partial} u(z)|^2\right)^{p/2} dA(z), \quad (6.86)$$

where $dA(z) := \frac{dx dy}{\pi}$, and $\partial, \bar{\partial}$ are the Wirtinger operators, that is

$$\partial := \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \bar{\partial} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right). \quad (6.87)$$

Proof. Since $\mathbb{C}_+ \sim \mathbb{R}_+^2$, every function $u = u(z) \in C_0^\infty(\mathbb{C}_+)$ can be seen as a function $u = u(x, y) \in C_0^\infty(\mathbb{R}_+^2)$. The function $\rho(x, y) := y \in W_{loc}^{1,p}(\mathbb{R}_+^2)$ is nonnegative in \mathbb{R}_+^2 , and $-\Delta_p \rho = 0$ on \mathbb{R}_+^2 . Moreover, $\frac{|\nabla \rho|^p}{\rho^{p-\alpha}} = \frac{1}{y^{p-\alpha}}$ and $\rho^\alpha = y^\alpha$ belong to $L_{loc}^1(\mathbb{R}_+^2)$. Then the hypotheses of Theorem 3.1 are fulfilled, hence we have:

$$\left(\frac{|p-1-\alpha|}{p}\right)^p \int_{\mathbb{R}_+^2} \frac{|u|^p}{y^{p-\alpha}} dx dy \leq \int_{\mathbb{R}_+^2} y^\alpha |\nabla u|^p dx dy, \quad u \in C_0^\infty(\mathbb{R}_+^2). \quad (6.88)$$

By the identification of \mathbb{R}_+^2 and \mathbb{C}_+ and the expressions of $\partial, \bar{\partial}$, we conclude the proof. \square

6.6 The Euclidean case

In this last section we show that our main results, Theorems 2.1 and 3.1, yield some well known sharp Hardy inequalities in the Euclidean space.

Since for $p < N$ \mathbb{R}^N is p -hyperbolic and is a Cartan-Hadamard manifold, Theorems 6.4 and 6.5 hold also on \mathbb{R}^N with $G_x = |x|^{\frac{p-N}{p-1}}$ and $r = |x|$ respectively. However, the function $|x|^{\frac{p-N}{p-1}}$ is p -harmonic in $\mathbb{R}^N \setminus \{0\}$ also for $p > N$. Therefore we have the following:

Theorem 6.8. *If $p \neq N$, then*

$$\left(\frac{|N - p| \cdot |p - 1 - \alpha|}{p(p - 1)} \right)^p \int_{\mathbb{R}^N \setminus \{0\}} |x|^{\alpha \frac{p-n}{p-1}} \frac{|u|^p}{|x|^p} dx \leq \int_{\mathbb{R}^N \setminus \{0\}} |x|^{\alpha \frac{p-n}{p-1}} |\nabla u|^p dx, \\ u \in C_0^\infty(\mathbb{R}^N \setminus \{0\}).$$

In the half space \mathbb{R}_+^N there holds the following:

Theorem 6.9. *Let $\alpha \in \mathbb{R}$, let $N \geq 2$, let $\mathbb{R}_+^N = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_1 > 0\}$, and let $\rho(x) := d(x, \partial\mathbb{R}_+^N)$ be the distance from the boundary of \mathbb{R}_+^N . Then we have*

$$\left(\frac{|p - 1 - \alpha|}{p} \right)^p \int_{\mathbb{R}_+^N} \rho^\alpha \frac{|u|^p}{\rho^p} dx \leq \int_{\mathbb{R}_+^N} \rho^\alpha |\nabla u|^p dx, \quad u \in C_0^\infty(\mathbb{R}_+^N). \quad (6.89)$$

Proof. The distance $\rho(x) = x_1 \in W_{loc}^{1,p}(\mathbb{R}_+^N)$ is nonnegative on \mathbb{R}_+^N and it is easy to verify that the hypotheses of Theorem 3.1 are satisfied. Therefore the thesis follows. \square

Actually a more general theorem for convex domains holds.

Theorem 6.10. *Let D be a proper convex open subset of \mathbb{R}^N , let $d := \text{dist}(\cdot, \partial D)$ be the distance from ∂D and let $\alpha < p - 1$. Then we have*

$$\left(\frac{p - 1 - \alpha}{p} \right)^p \int_D d^\alpha \frac{|u|^p}{d^p} dx \leq \int_D d^\alpha |\nabla u|^p dx, \quad u \in C_0^\infty(D),$$

and, in particular,

$$\left(\frac{p - 1}{p} \right)^p \int_D \frac{|u|^p}{d^p} dx \leq \int_D |\nabla u|^p dx, \quad u \in C_0^\infty(D).$$

Proof. The thesis will follow by applying Theorems 2.1 and 3.1. To this end it suffices to prove that the function $d(x) := \text{dist}(x, \partial D)$ is p -superharmonic. Indeed, $D = \bigcap \Pi$, and $d(x) = \inf_{\Pi} \text{dist}(x, \partial \Pi)$ where the intersection and the infimum are taken over all the half-spaces Π containing D . Since $\text{dist}(x, \partial \Pi)$ is continuous and p -harmonic, we have that d is p -superharmonic (see [HKM]). This concludes the proof. \square

A Appendix

Let us recall that p -capacity of a compact set K is defined as

$$\text{cap}_p(K, M) = \inf \left\{ \int_M |\nabla u|^p dv_g : u \in C_0^\infty(M), 0 \leq u \leq 1, \right. \\ \left. u = 1 \text{ in a neighborhood of } K \right\}. \quad (\text{A-1})$$

Proposition A-1. *Let M be a p -hyperbolic manifold of dimension N . Let $K \subset M$ be a compact set of zero p -capacity. Then*

$$D^{1,p}(M) \subset D^{1,p}(M \setminus K),$$

that is every function $u \in D^{1,p}(M)$ can be approximated by function $C_0^\infty(M \setminus K)$ in the norm $|\cdot|_{D^{1,p}}$.

Proof. Let $\varphi \in C_0^\infty(M)$. In order to prove the claim it is sufficient to prove that $\varphi \in D^{1,p}(M \setminus K)$. Since $\text{cap}_p(K, M) = 0$, there exists a sequence $(u_j)_{j \geq 1}$ such that, for any $j \geq 1$, $u_j \in C_0^\infty(M)$, $0 \leq u_j \leq 1$, $u_j = 1$ in a neighborhood of K and $u_j \rightarrow 0$ in $D^{1,p}(M)$. For every $j \geq 1$ the function $\varphi_j := (1 - u_j)\varphi$ belongs to $C_0^\infty(M \setminus K) \subset D^{1,p}(M \setminus K)$. We shall prove that $\varphi_j \rightarrow \varphi$ in $D^{1,p}(M \setminus K)$, that is

$$\int_{M \setminus K} |\nabla \varphi_j - \nabla \varphi|^p dv_g \longrightarrow 0, \quad (\text{as } j \rightarrow +\infty). \quad (\text{A-2})$$

In fact, we have

$$\begin{aligned} \left(\int_{M \setminus K} |\nabla \varphi_j - \nabla \varphi|^p dv_g \right)^{1/p} &= \left(\int_{M \setminus K} |\nabla \varphi (1 - u_j) - \varphi \nabla u_j - \nabla \varphi|^p dv_g \right)^{1/p} \\ &\leq \left(\int_{M \setminus K} |\nabla \varphi|^p |u_j|^p dv_g \right)^{1/p} + \left(\int_{M \setminus K} |\varphi|^p |\nabla u_j|^p dv_g \right)^{1/p}. \end{aligned} \quad (\text{A-3})$$

The second term in (A-3) converges to 0 for $j \rightarrow +\infty$. Indeed, since $u_j \rightarrow 0$ in $D^{1,p}(M)$, we obtain

$$\begin{aligned} \int_{M \setminus K} |\varphi|^p |\nabla u_j|^p dv_g &\leq \int_M |\varphi|^p |\nabla u_j|^p dv_g \\ &\leq |\varphi|_\infty^p \int_M |\nabla u_j|^p dv_g \longrightarrow 0, \quad (\text{as } j \rightarrow +\infty). \end{aligned}$$

It remains to prove that the first term in (A-3) converges to 0 as well. Let D be the support of φ ; then we get

$$\begin{aligned} \int_{M \setminus K} |\nabla \varphi|^p |u_j|^p dv_g &\leq \int_M |\nabla \varphi|^p |u_j|^p dv_g = \int_D |\nabla \varphi|^p |u_j|^p dv_g \\ &\leq |\nabla \varphi|_\infty^p \int_D |u_j|^p dv_g \leq |\nabla \varphi|_\infty^p C \int_M |\nabla u_j|^p dv_g \longrightarrow 0, \quad (\text{as } j \rightarrow +\infty), \end{aligned}$$

where, in the last inequality, we have used a characterization of the p -hyperbolic manifold (see Theorem 3 in [Tr2]). \square

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